Image Analysis as an Inverse Problem: Metrics, Regularization, and Geometric Analysis

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Mathematical Modeling and Analysis (T-7)

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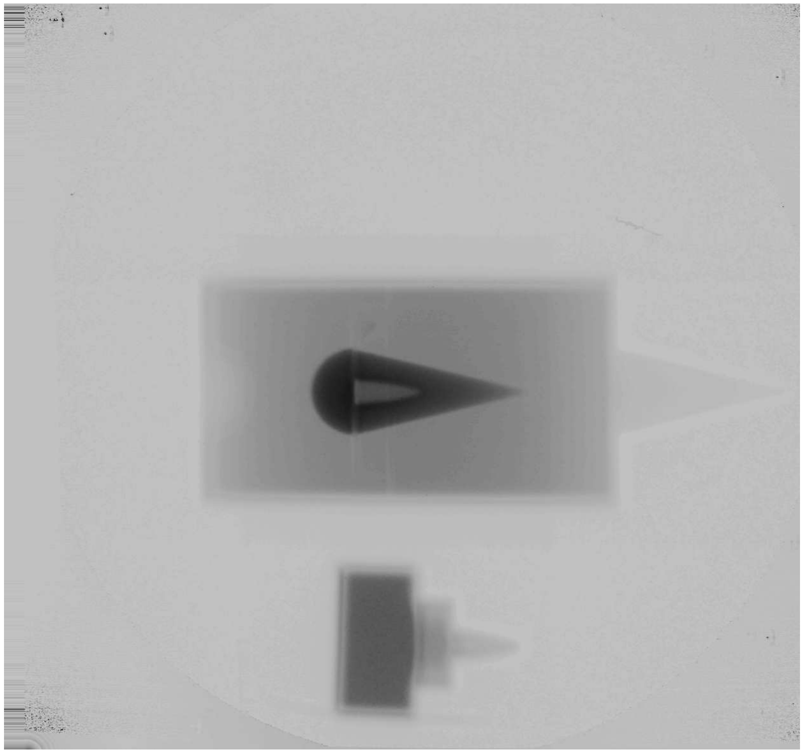
http://ddma.lanl.gov

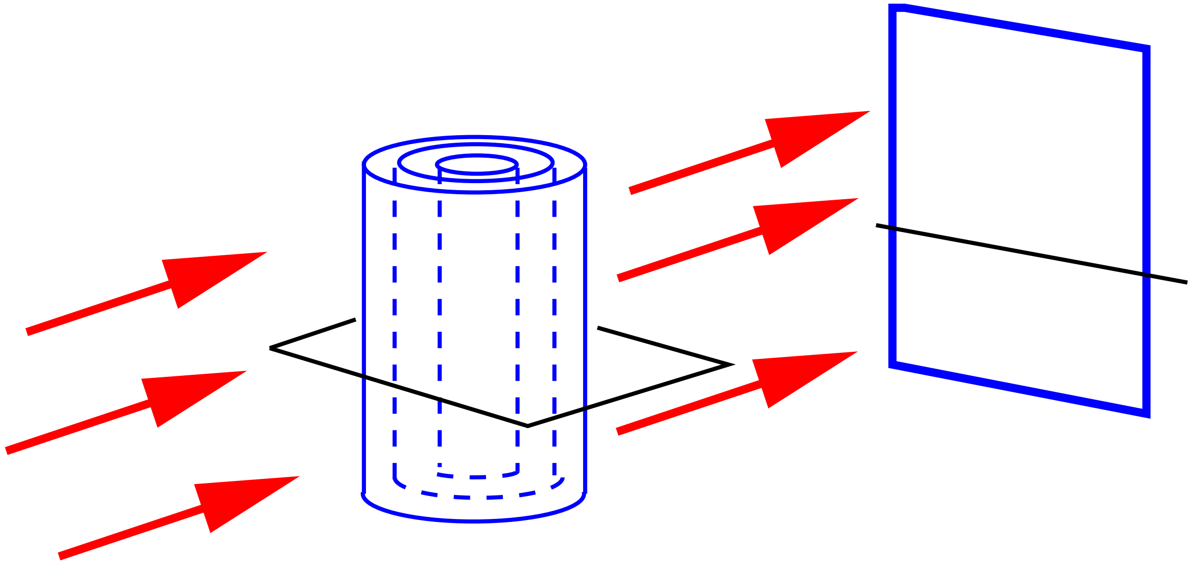
Los Alamos
NATIONAL LABORATORY

Images Tasks: Face Recognition

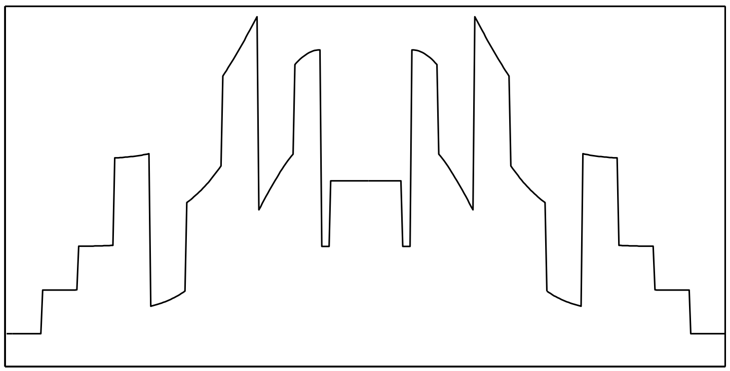
Images Tasks: Sparse Tomography



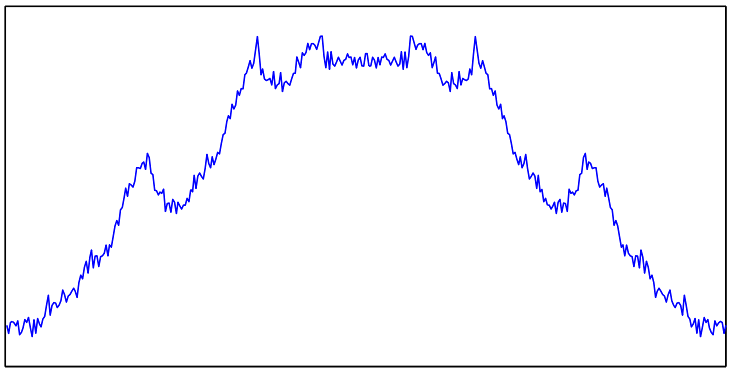
Images Tasks: One View Tomography with Symmetry



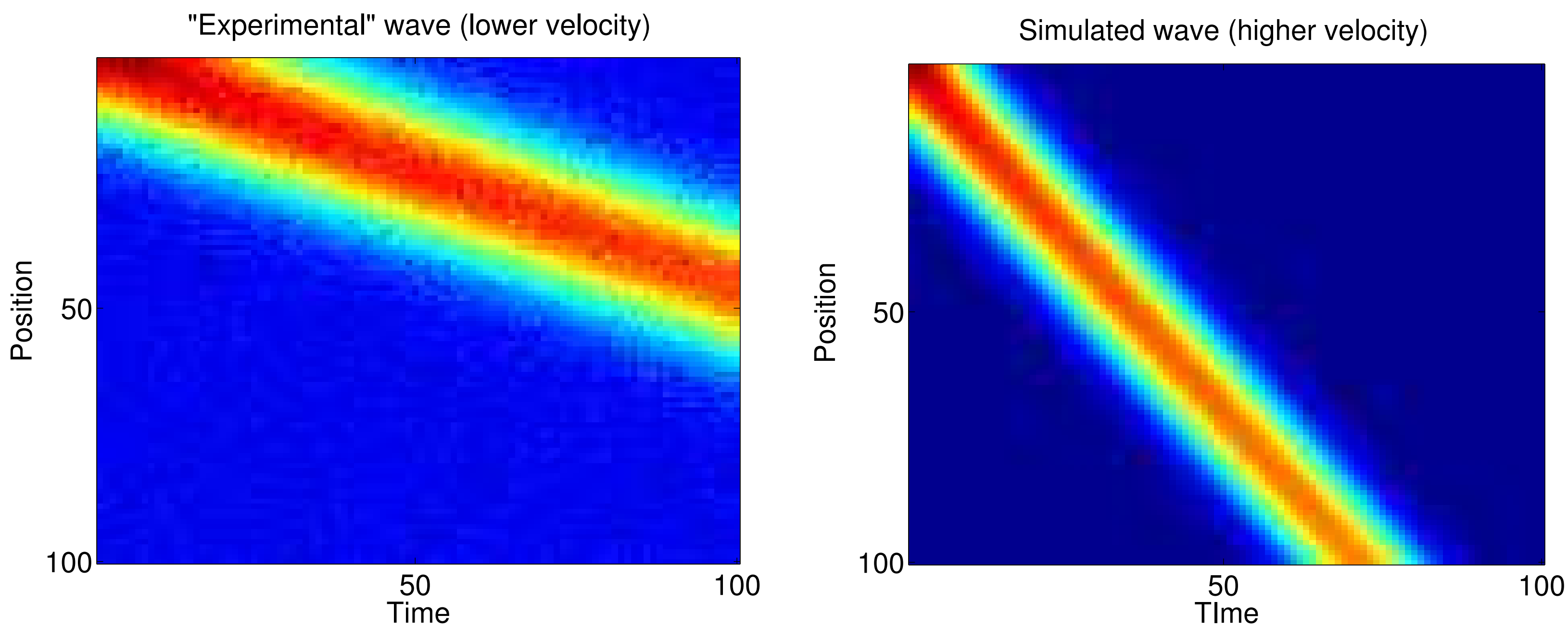
Object Radial Density Profile



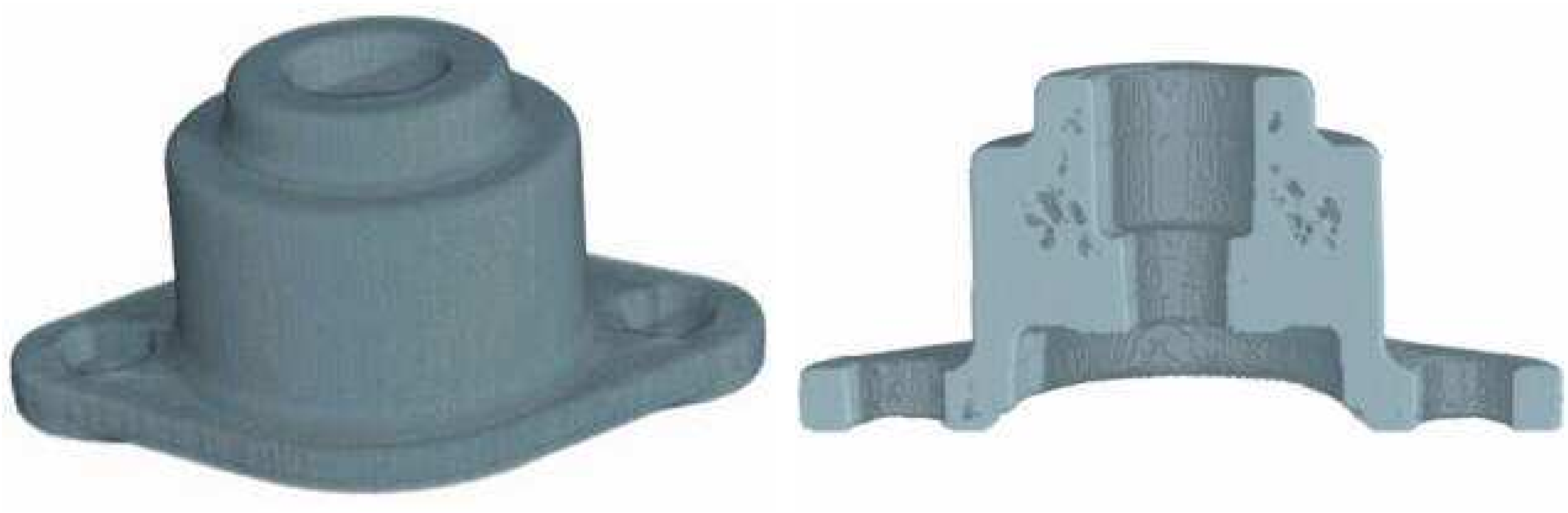
Noisy Areal Density Profile (Mass Projection)



Images Tasks: Matching Experiments and Data



Images Tasks: Material Damage assessment with X-rays



(from M. Simon and C. Sauerwein's paper in `http://www.ndt.net/article/wcndt00/`)

Inverse Problems and Notation

- A measured image f
- A reconstructed or ideal image or density distribution u

Task: Find u given f

$$
\begin{aligned}
f = u + \eta &\quad\rightarrow\quad \text{additive noise}\\
f = F(u) &\quad\rightarrow\quad \text{nonlinear/stochastic transformation}\\
f = P(u) &\quad\rightarrow\quad \text{measurement operator}\\
f = P(u) + \eta &\quad\rightarrow\quad \text{measurement operator + noise}\\
f = F(P(u) + \eta) &\quad\rightarrow\quad \text{everything}
\end{aligned}
$$

A Probabilistic Framework = Variational Formulation

$$
\begin{aligned}
p(u|f) &\sim p(f|u)p(u) \\
\arg\max_u p(u|f) &= \arg\max_u p(f|u)p(u) \\
\arg\min_u \{-\log(p(u|f))\} &= \arg\min_u \{-\log(p(f|u)) - \log(p(u))\}
\end{aligned}
$$

Suppose that

$$
\begin{aligned}
p(f|u) &\sim e^{-\int \lambda|u-f|^2} \\
p(u) &\sim e^{-\int |\nabla u|}
\end{aligned}
$$

We arrive at:

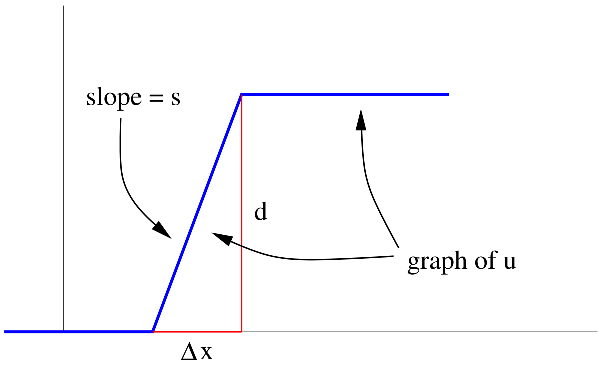
$$\arg\min_u \left\{ \int |\nabla u| + \lambda \int |u-f|^2 \right\}.$$

More generally, with $p(f|u) \sim e^{-\lambda\rho(u,f)}$ and $p(u) \sim e^{-E(u,\nabla u)}$, we get:

$$\arg\min_u \{E(u,\nabla u) + \lambda\rho(u,f)\}.$$

Regularizations are prior models: $TV(u) = \int |\nabla u|$

Consider $F(u) \equiv \int |\nabla u|^p dx$



$$F(u) = s^p(\Delta x) = \frac{(s\Delta x)^p}{(\Delta x)^{p-1}} = d^p(\Delta x)^{1-p}$$

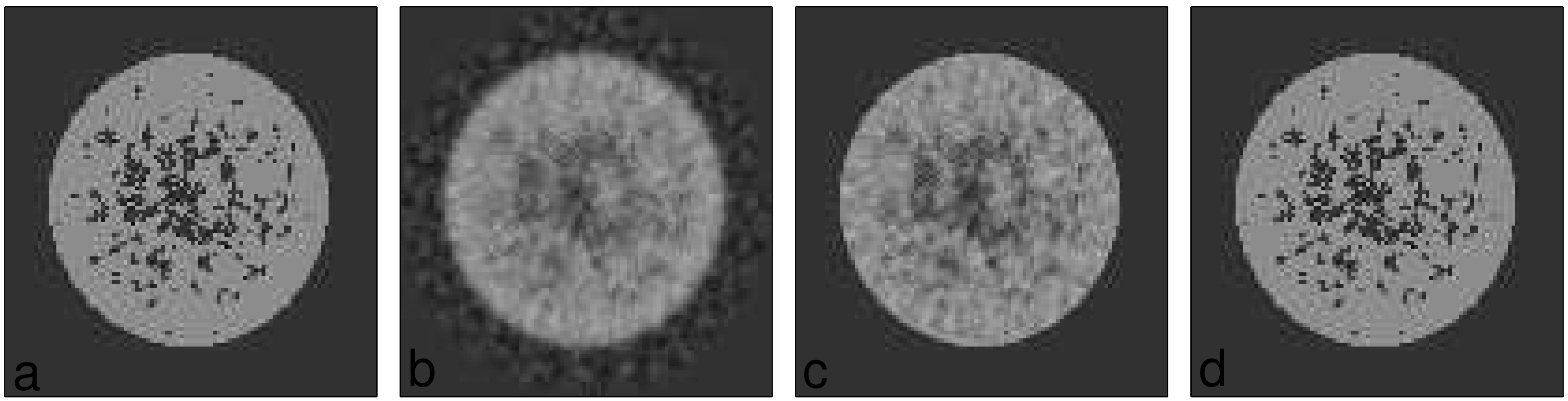
$(p > 1)$ $F(u) \underset{\Delta x \to 0}{\longrightarrow} \infty$ discontinuities are avoided: smooth u preferred,

$(p < 1)$ $F(u) \underset{\Delta x \to 0}{\longrightarrow} 0$ discontinuities cost nothing: step u preferred,

$(p = 1)$ $F(u) = d$ only jump magnitude “counts”, no bias towards smooth or step.

Getting Priors Right is Useful: Material Damage

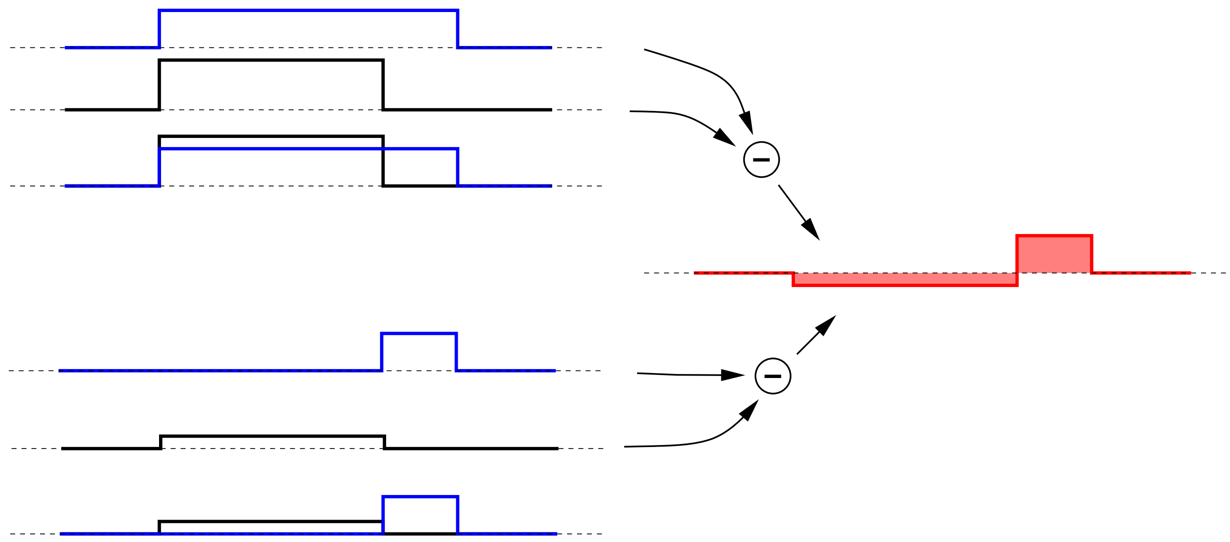
Simple illustration of the power of proper priors in the case of simulated x-ray radiographic measurement of material damage:



a) Simulated damage, b) SVD, c) Non-negative SVD, d) binary prior SVD.

Metrics for Comparisons: $p(f|u) \Rightarrow$ metric $\rho(u, f)$

Typical metrics are *norms* of differences, $\rho(u, f) = ||u - f||$. This can have undesirable effects.



We want metrics which in effect split differences nonlinearly and weight the factors differently.

Metrics: Splitting Differences

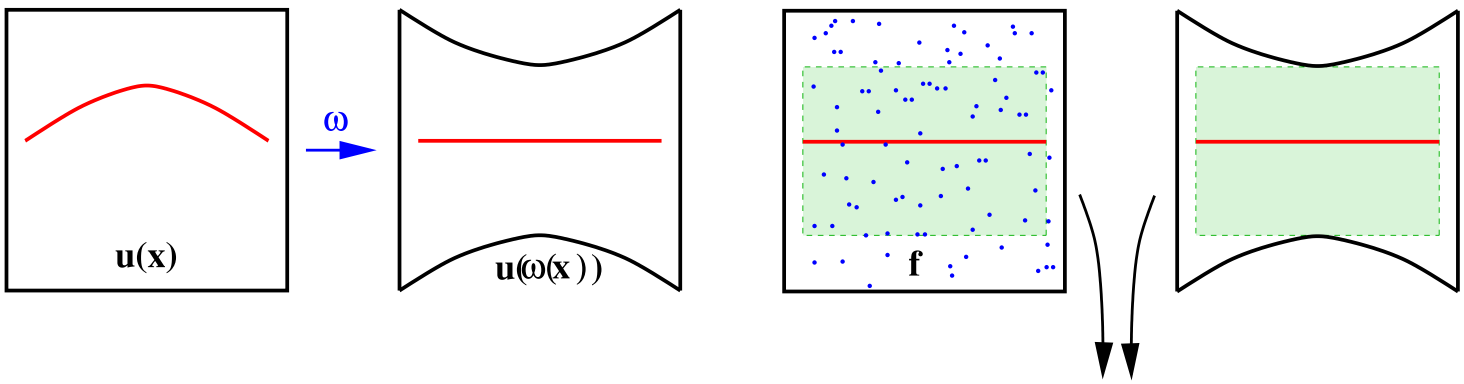
We want metrics which care more about some directions than others or at least measure them differently:

- Warps: Metrics which split the differences nonlinearly and measure components differently
- Quotients: Metrics which split the differences nonlinearly ignore one component.

Warps: Metrics Separating Noise and Model Error components.

Idea: Warp domain to match image u and f. Use a natural stochastic term $\log(p_n())$ to measure remaining difference.

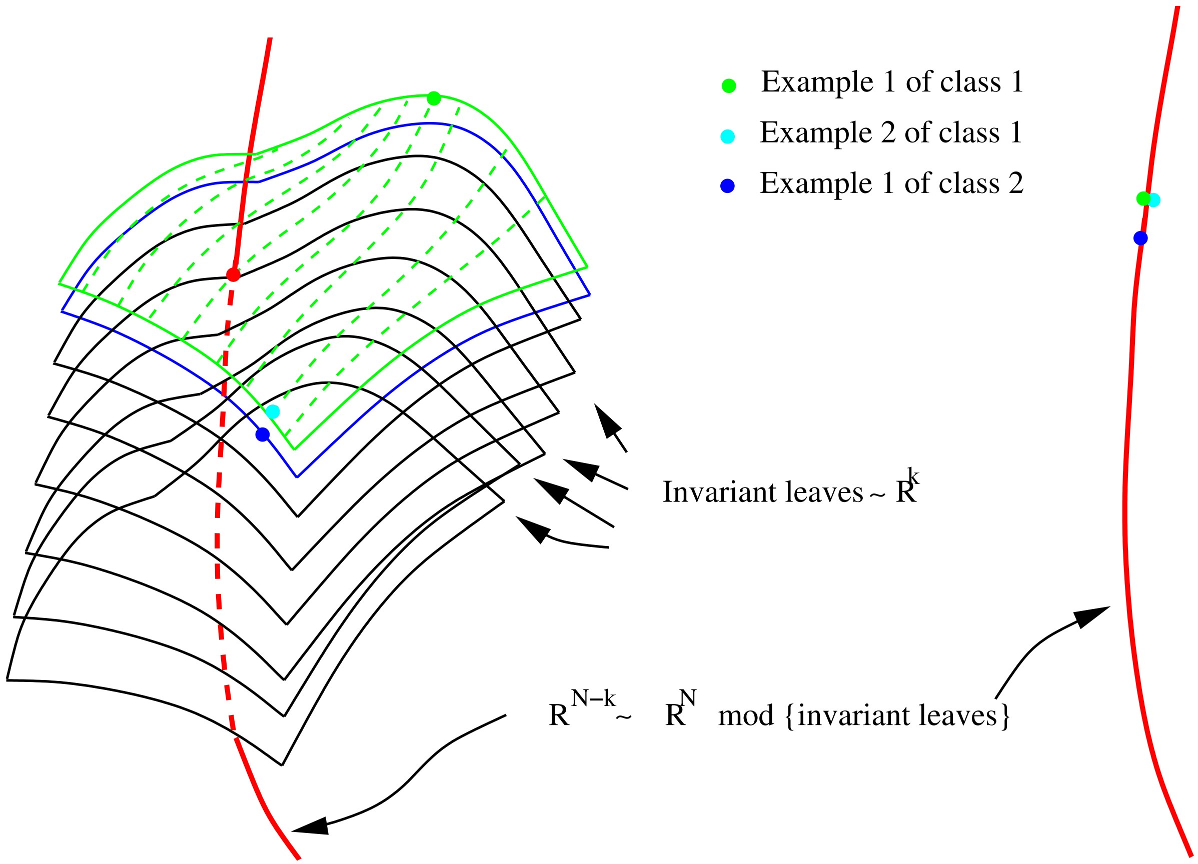
$$\rho(u(x), f) = E(\omega) - \alpha \log(p_n(f|u(\omega(x))))$$



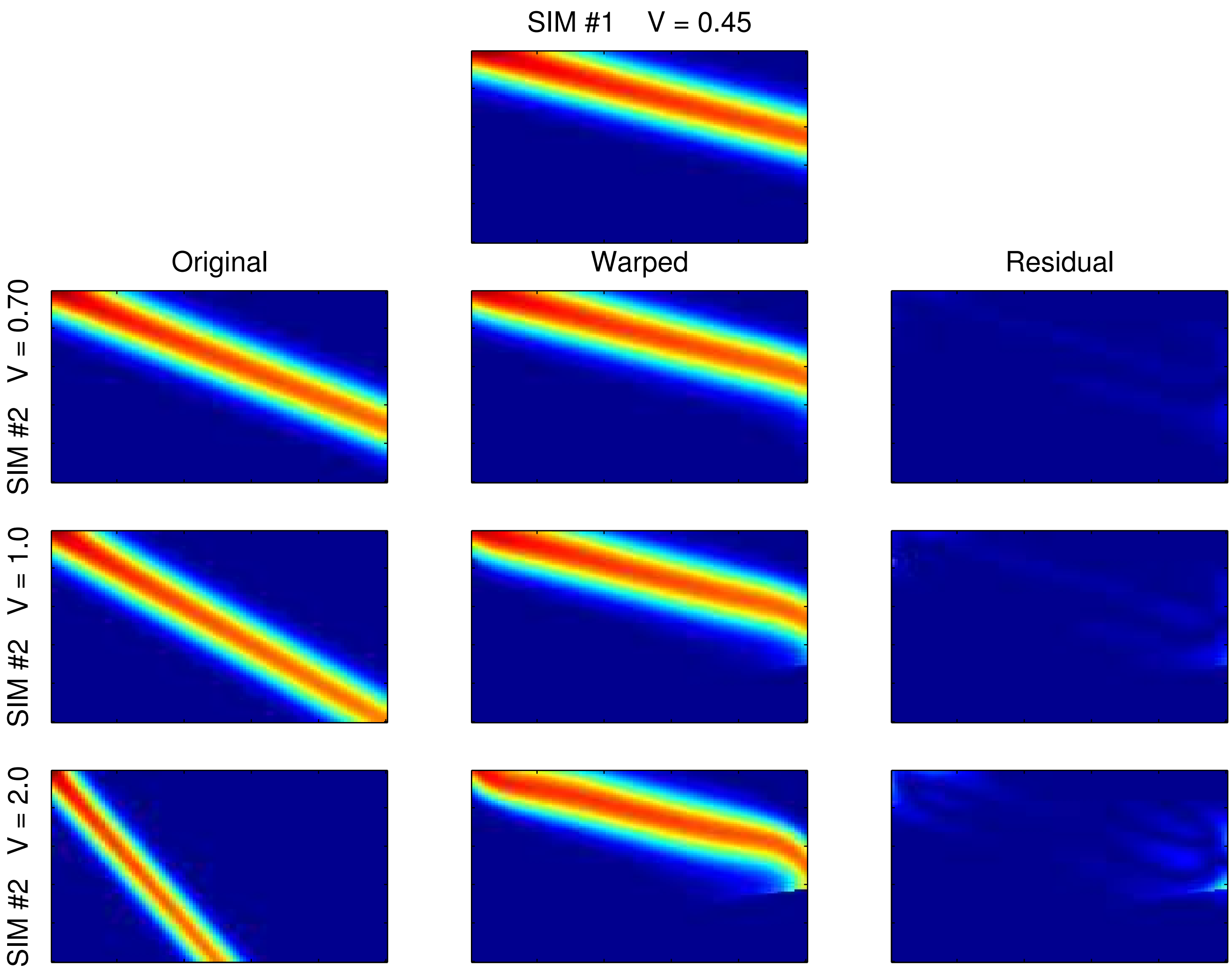
$$\rho(\mathbf{u}, \mathbf{f}) = \mathbf{E}(\omega(\mathbf{x})) - \alpha\ \mathbf{log}(\mathbf{p}_n(\mathbf{f}\ |\ \mathbf{u}(\omega(\mathbf{x}))))$$

(on green comparison region)

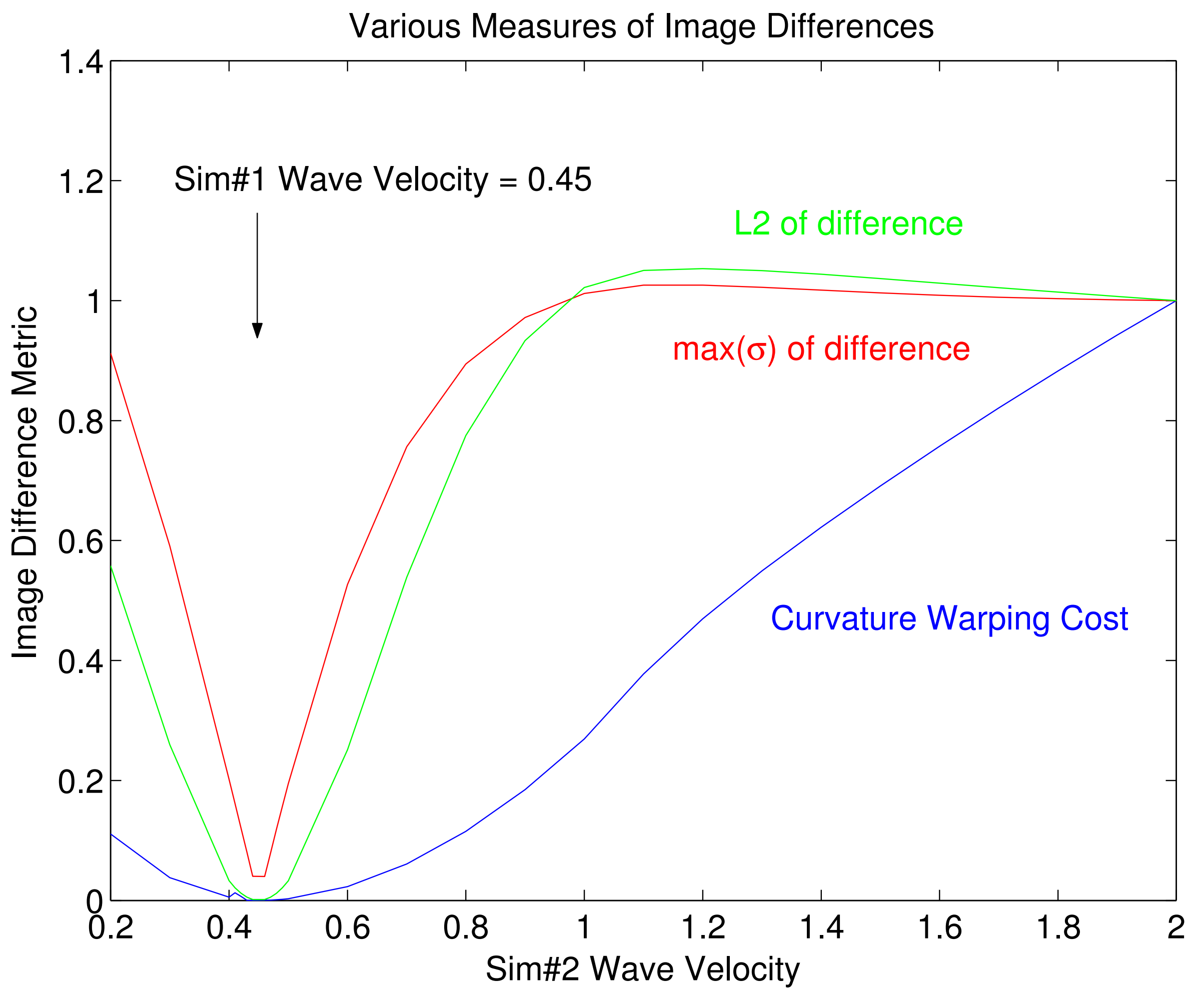
Quotients: Metrics that Ignore Unimportant Differences.



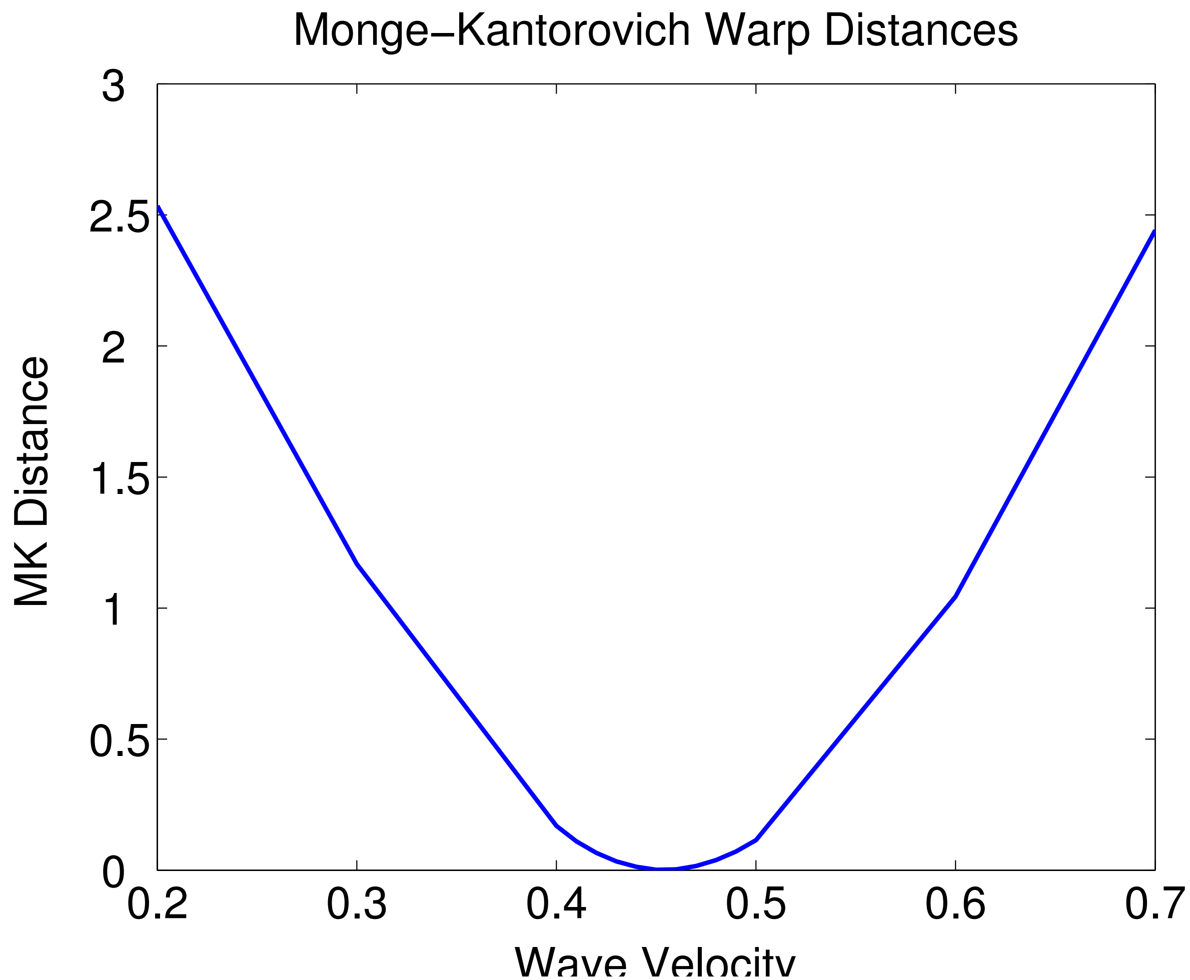
Metrics: Fluid Warping Example



Metrics: Fluid Warping Example

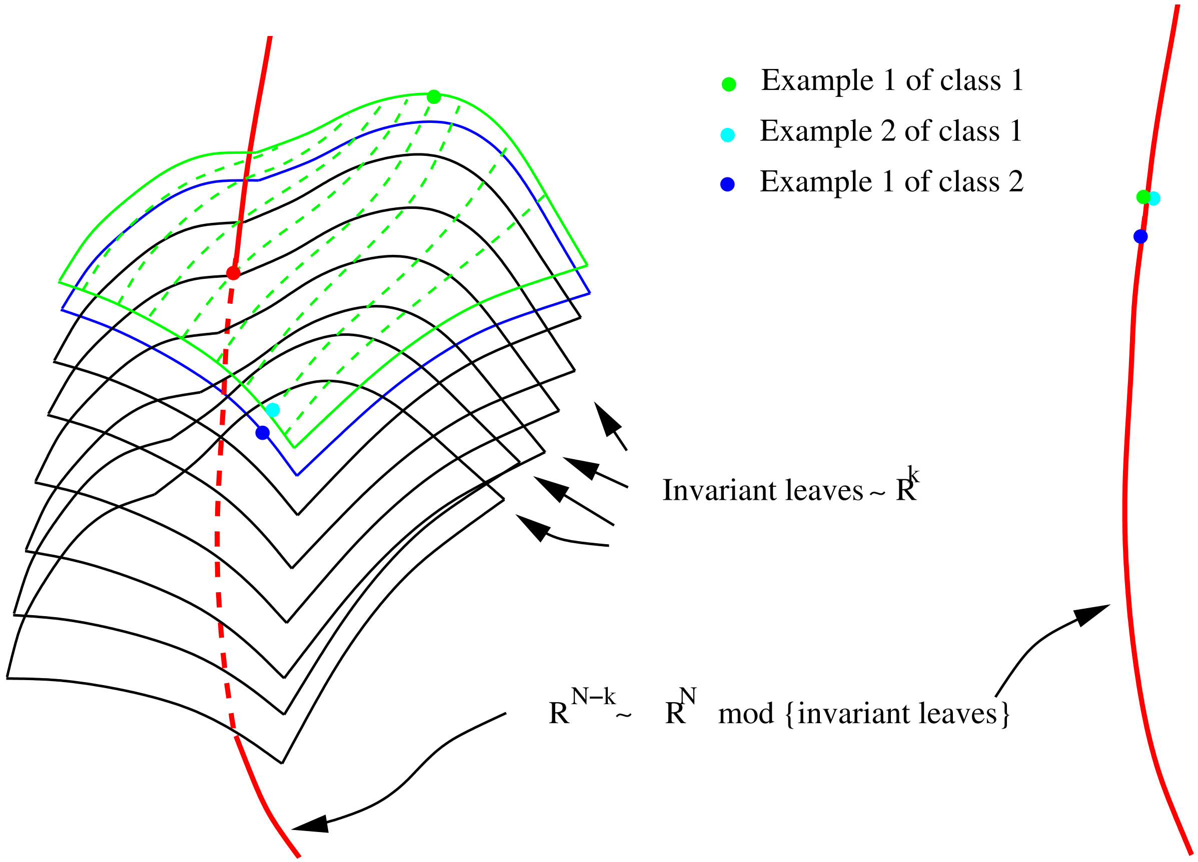


Metrics: MK Warping Example



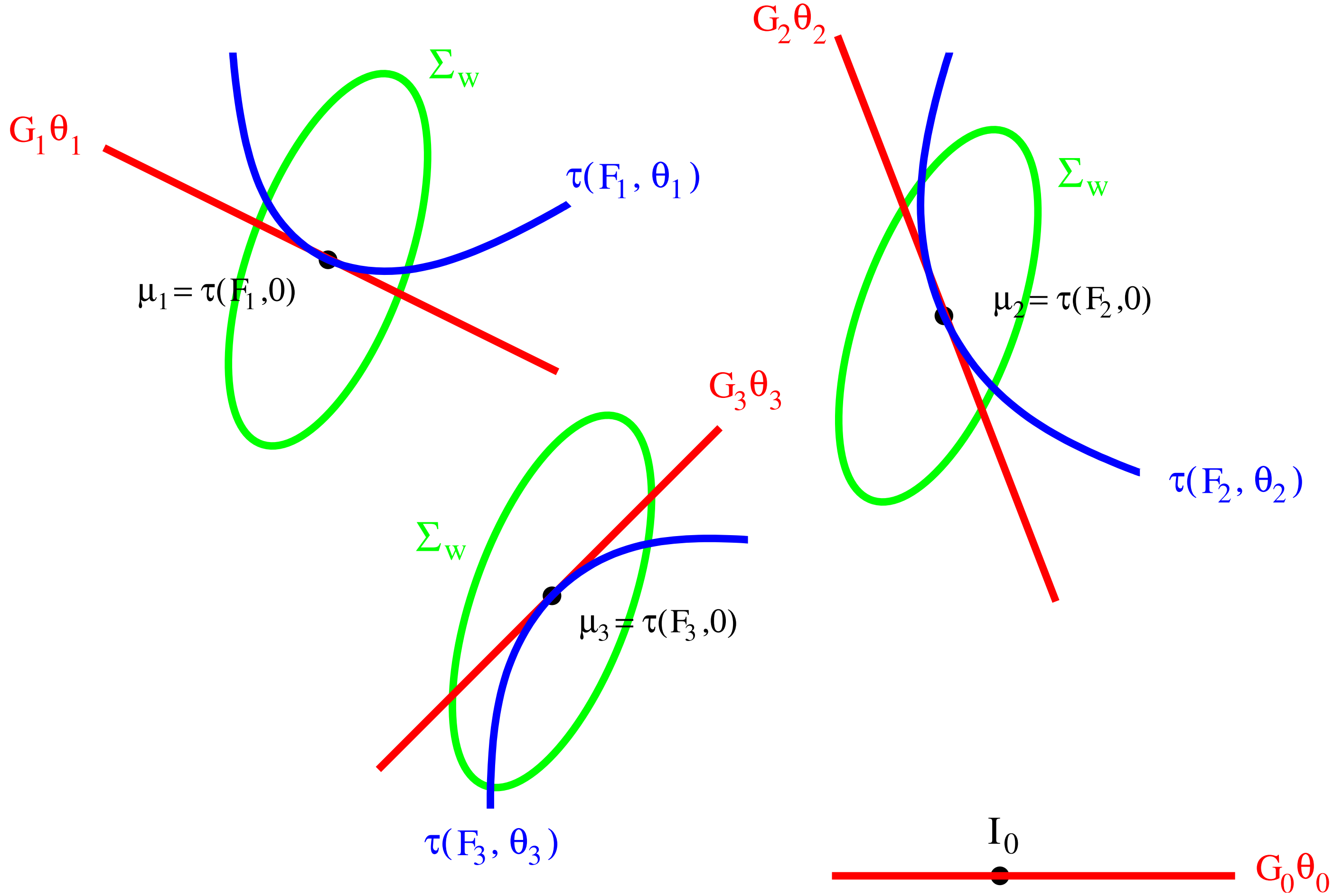
The classic Monge-Kantorovich problem applied to image warping.

Metrics: Classification Mod Invariance Applied to Faces



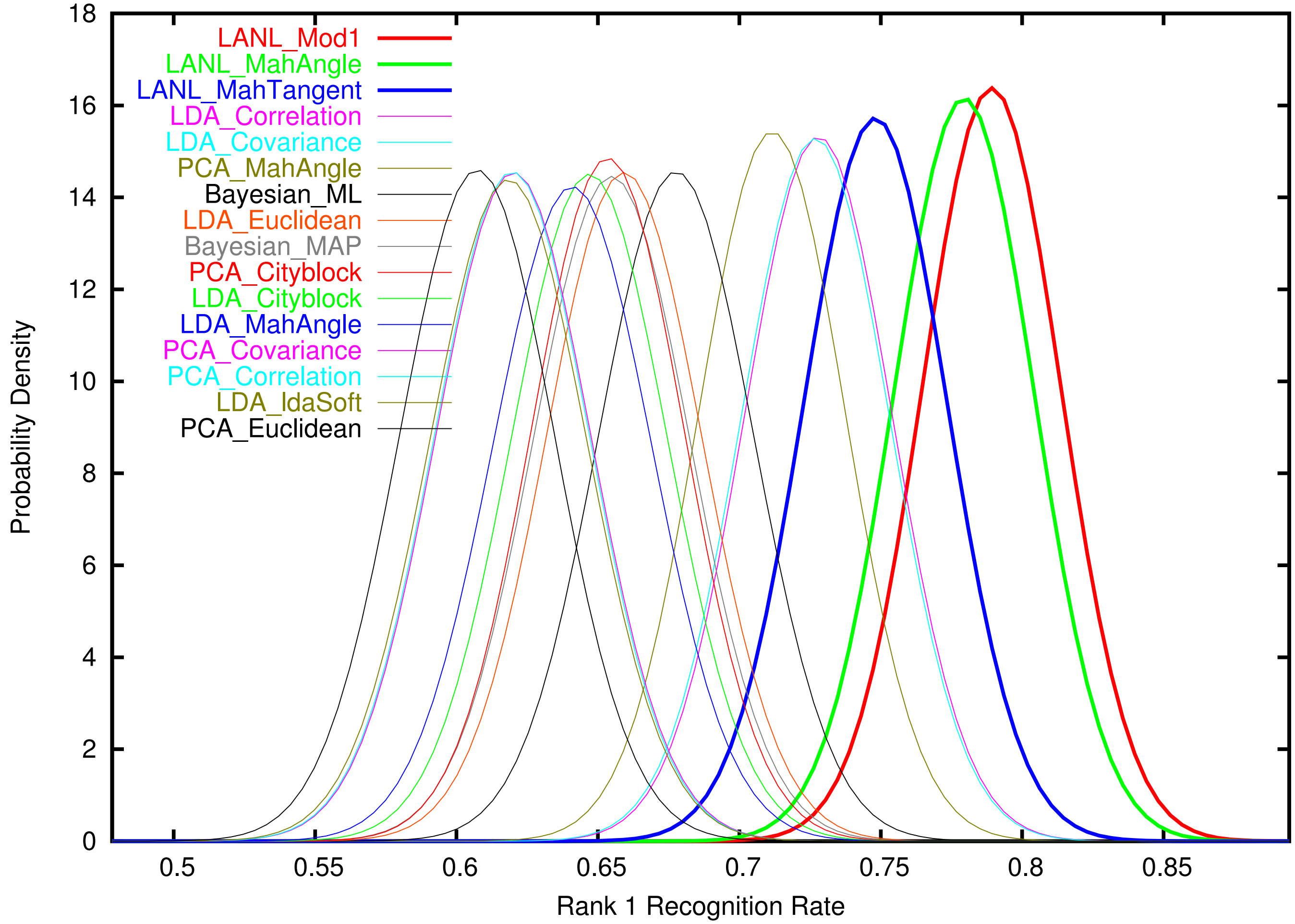
What we would like to do: factor out the orbits of transformations to which we desire invariance.

Classification Mod Invariance: Our Novel Approach:



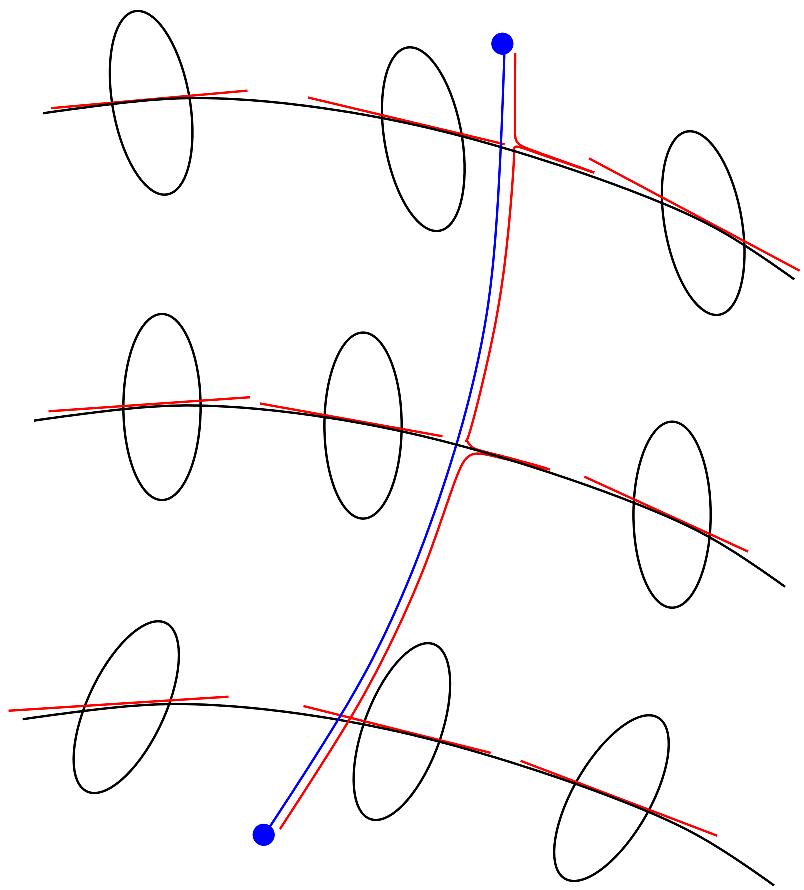
How we get conditional measures which approximate the quotient space metrics.

Classification Mod Invariance: Face Recognition Test



Our results (the three rightmost curves) are quite good!

Geodesics for Singular Riemannian Metrics



Replace $M \rightarrow PP^T MPP^T + \alpha I$ then let $\alpha \rightarrow 0$.

Total Variation Reconstruction: L^2TV

In the early 1990's Rudin, Osher and Fatemi suggested using a total variation term for regularization of the image restoration problem.

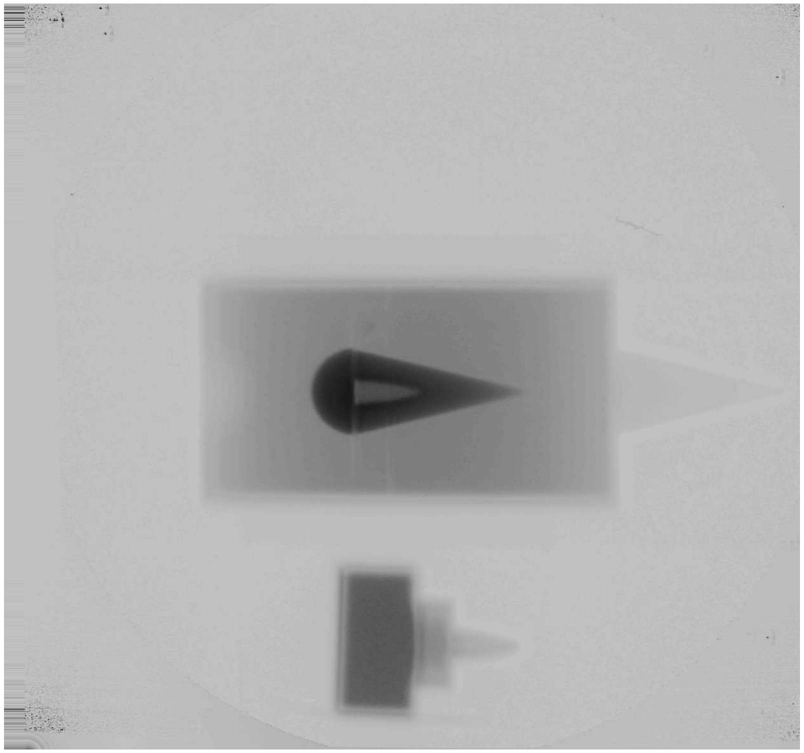
$$\begin{aligned} \min_u \ F(u) &= \int |\nabla u| + \lambda \int |u-f|^2 && (1) \\ &= TV(u) + \lambda ||u-f||_2^2 && (2) \end{aligned}$$

In addition to a large amount of intriguing and beautiful theory to be explored this and similar functionals are big improvements on previously used methods. In the following, we use:

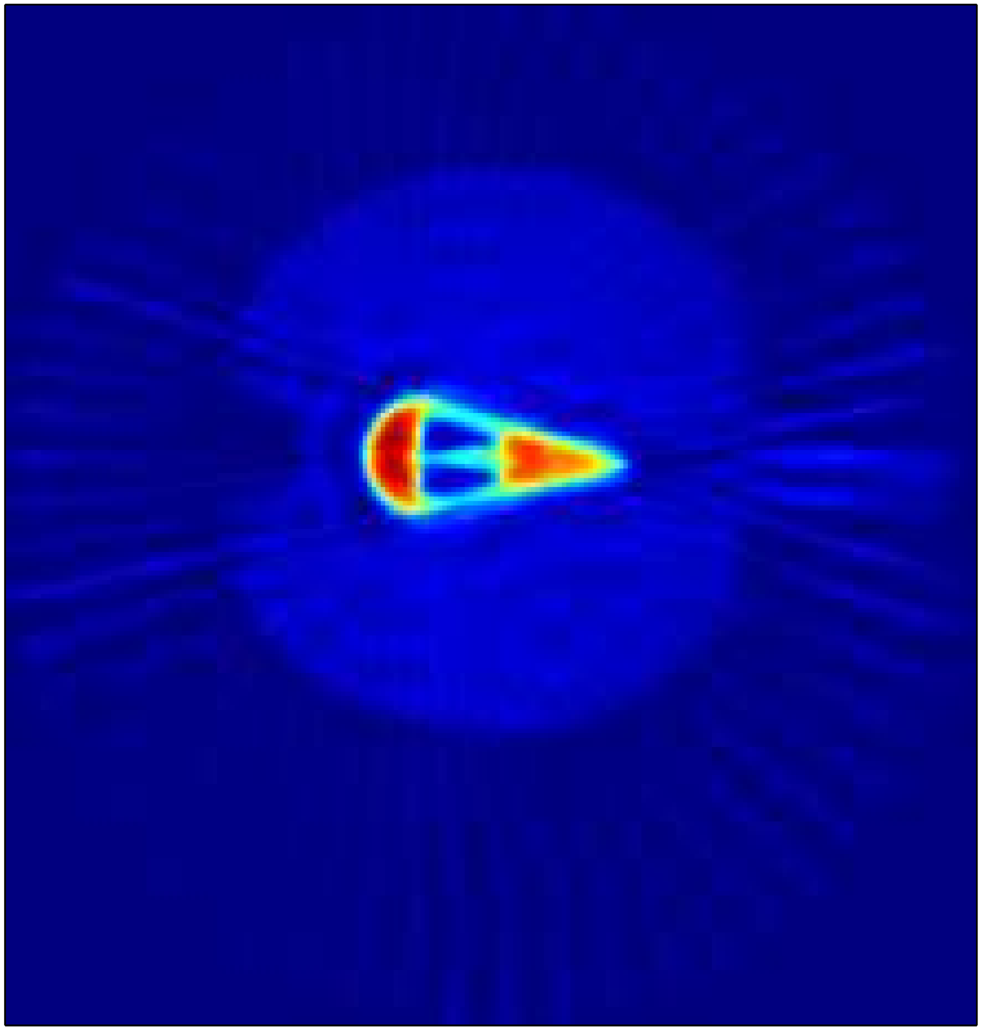
$$\min_u \ F(u) = \int |\nabla u| + \lambda \int |P(u) - f|^2$$

Example: BCO4

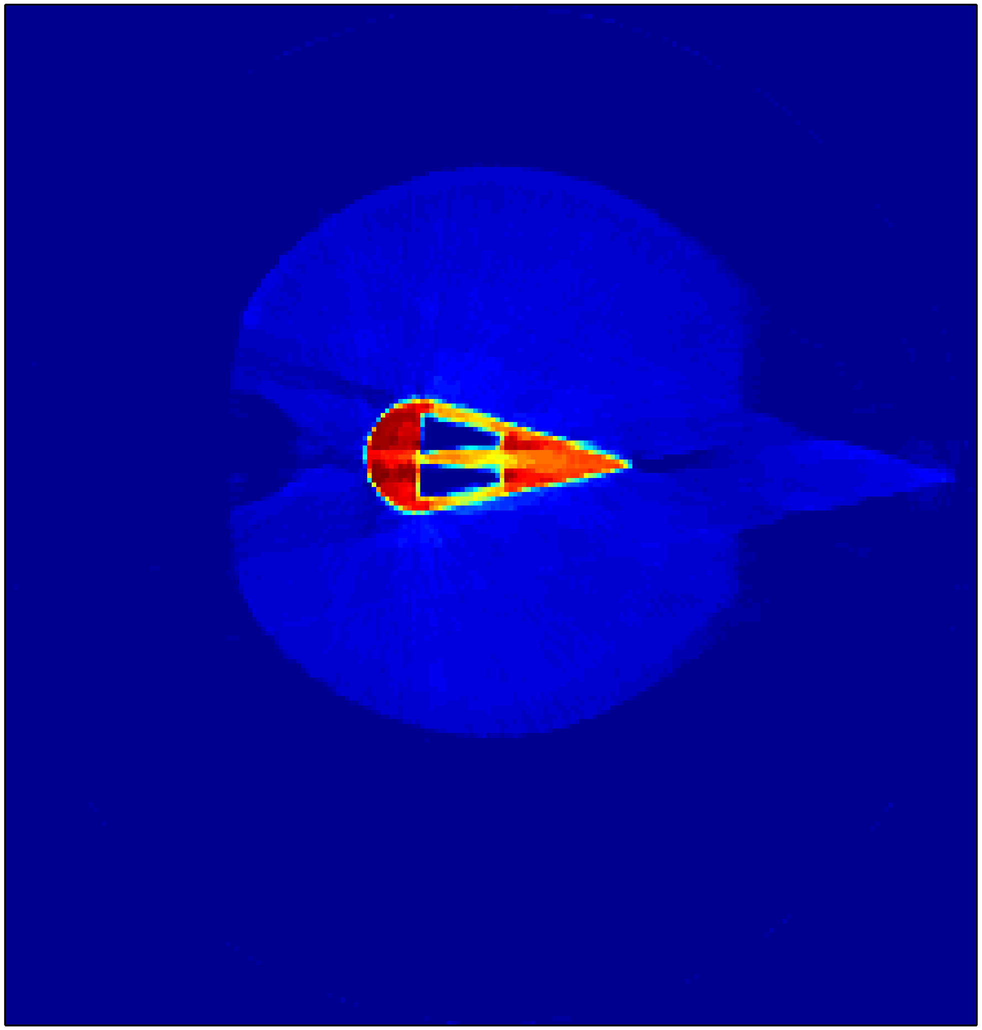
The multiple view test object: a proton radiograph from one of 30 viewing angles. The data was collected at the Los Alamos Neutron Science Center (LANSCE) in the proton radiography facility. Thirty viewing angles were used to collect tomographic data.



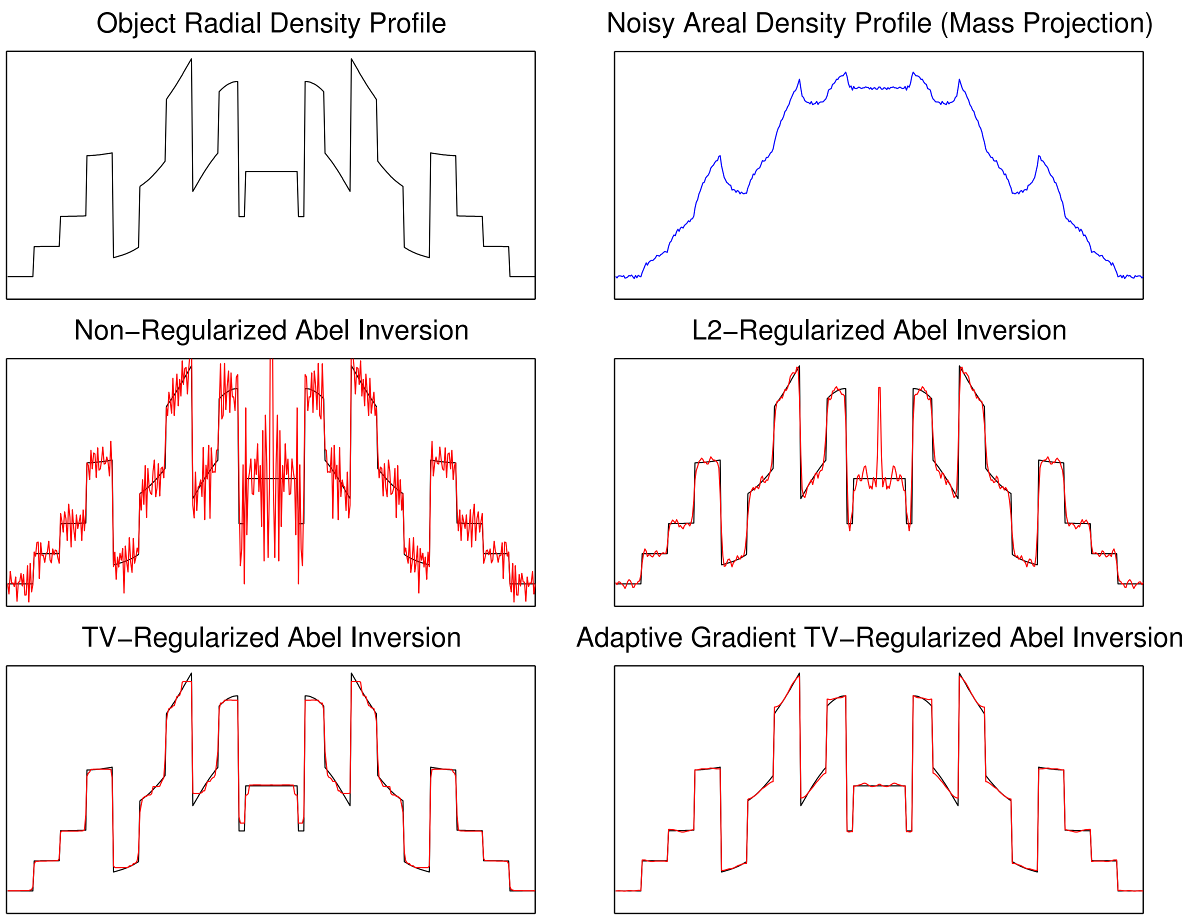
Example: BCO4, SVD regularized



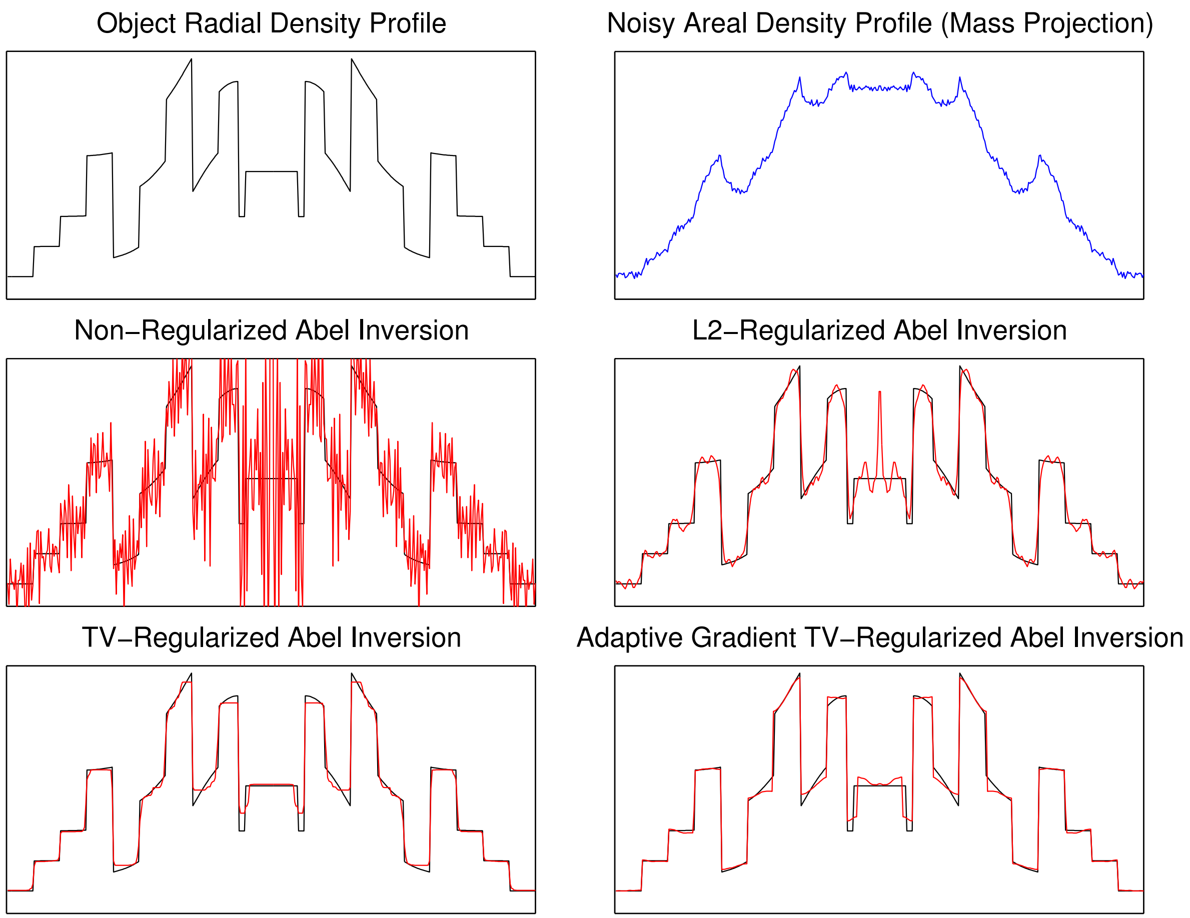
Example: BCO4, TV regularized



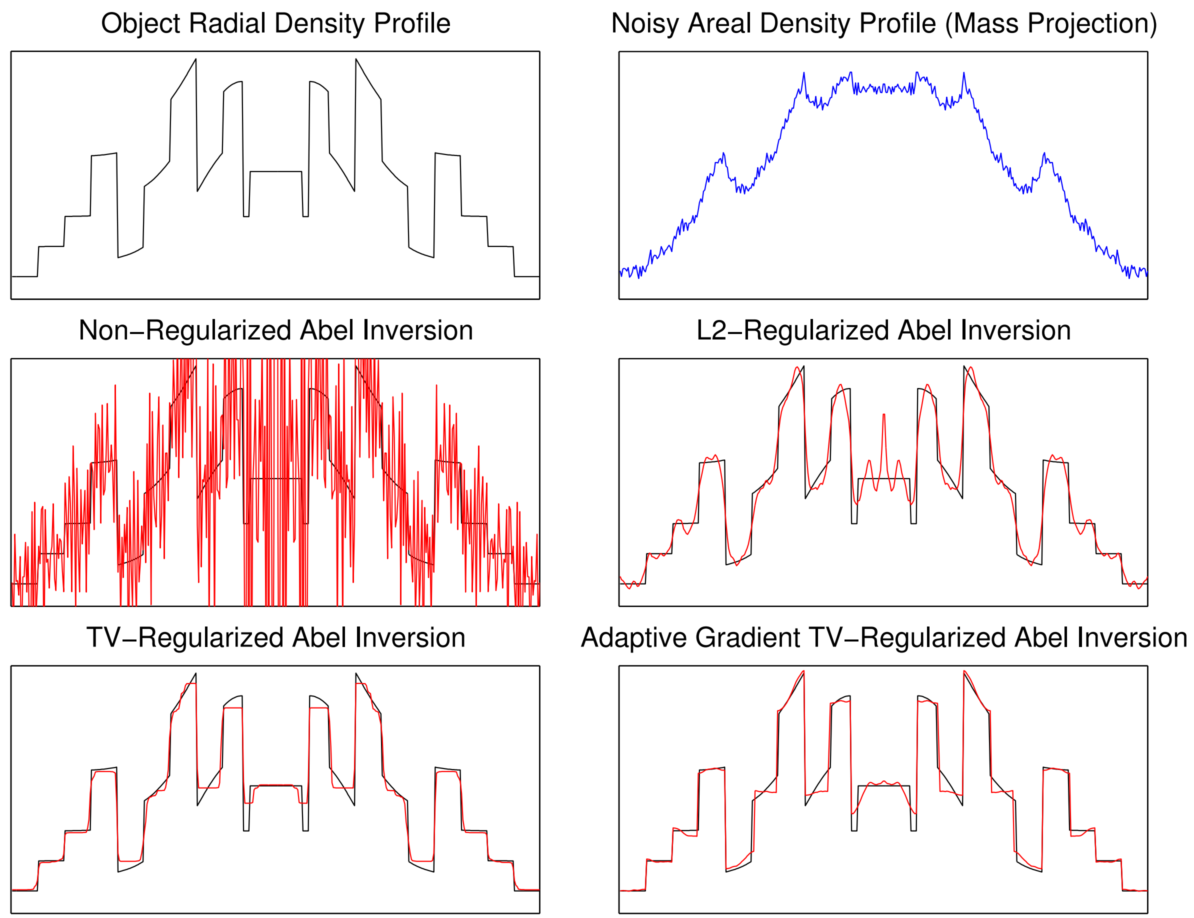
Example: TVAbel



Example: TVAbel



Example: TVAbel



Geometric Analysis: L^1TV

How are image analysis/processing methods really applied?

- Real, discrete, noisy data
- Approximate computations
- Convergence guessed at

Most of the time there is little ability to understand precisely how solutions relate to exact solutions.

When available, non-trivial exact solutions can play a very important role in understanding and evaluating a method.

With this in mind, we look more closely at total variation minimization: here we have a peek at more mathematical detail and a sample result.

Geometric Analysis: $L^1 TV$

If we replace L^2 fidelity with L^1 fidelity we get the L^1 total variation functional (L^1TV):

$$\min_u \; F(u) = \int |\nabla u| + \lambda \int |u - f|$$

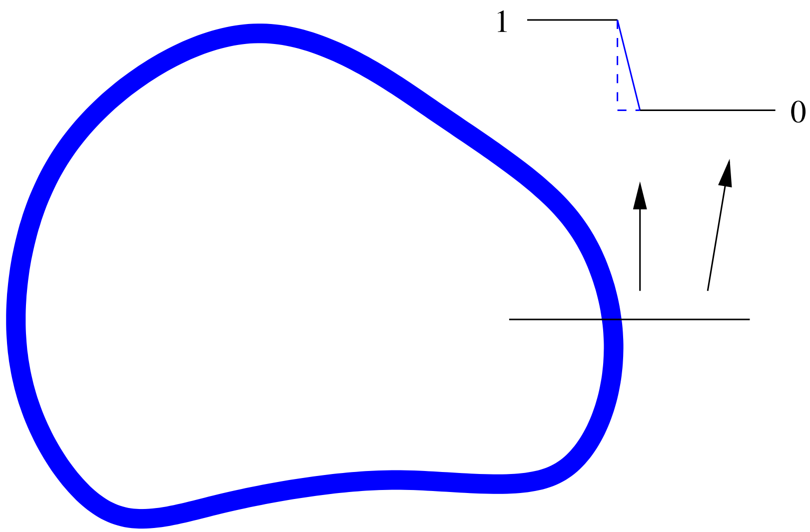
or

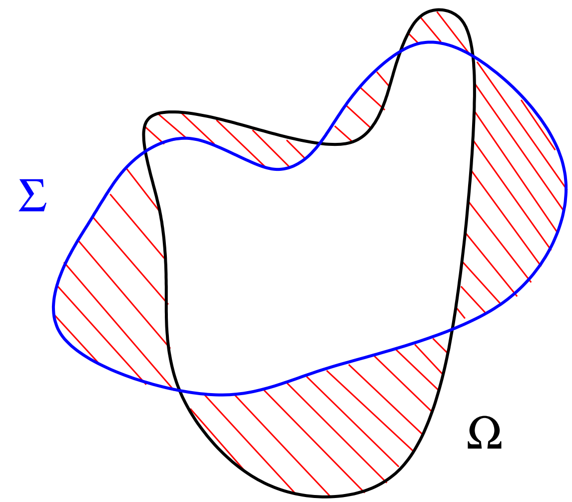
$$\min_u \; F(u) = \int |\nabla u| + \lambda \int |P(u) - f|$$

This equation is very interesting from a geometric viewpoint.

Geometric Analysis: $L^1 TV$

- $u = \chi_\Sigma \rightarrow \int |\nabla u|$ = perimeter of Σ
- $u = \chi_\Sigma, f = \chi_\Omega \rightarrow \lambda \int |u - f| = \lambda \int |\chi_\Sigma - \chi_\Omega| = \lambda \text{Area}(\Sigma \bigtriangleup \Omega)$



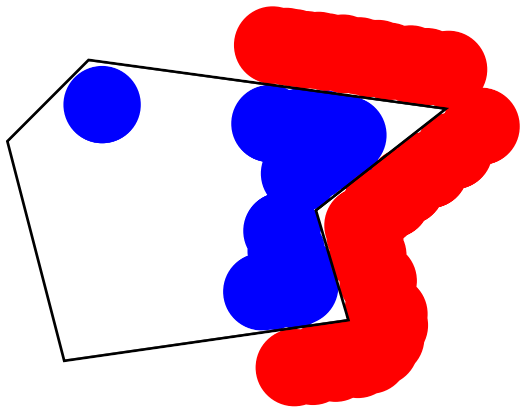


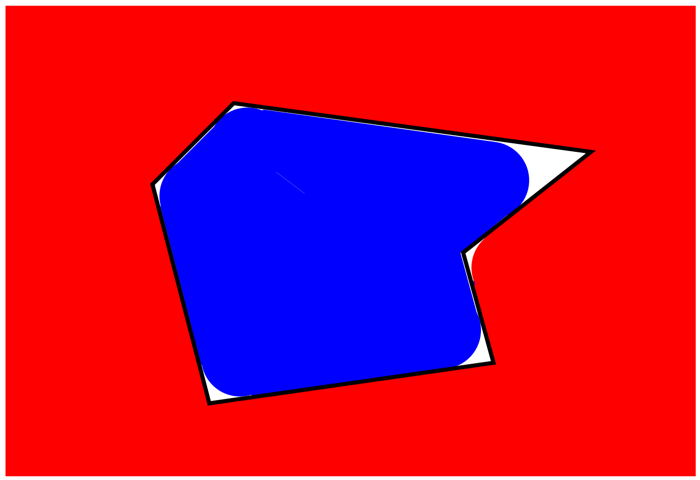
$f = \chi_\Omega \rightarrow u = \chi_\Sigma$ and $F(u) = F(\Sigma) = \text{Per}(\Sigma) + \lambda \text{Area}(\Sigma \bigtriangleup \Omega)$

Geometric Analysis: L^1TV, $B_{\frac{2}{\lambda}} \subset \Omega \rightarrow B_{\frac{2}{\lambda}} \subset \Sigma$

Theorem 1. *If $B_r \subset \Omega$ where $r \geq \frac{2}{\lambda}$, then $B_r \subset \Sigma$.*

In particular, we can conclude that the boundary of Σ is in the envelope of inside and outside $\frac{2}{\lambda}$ balls.

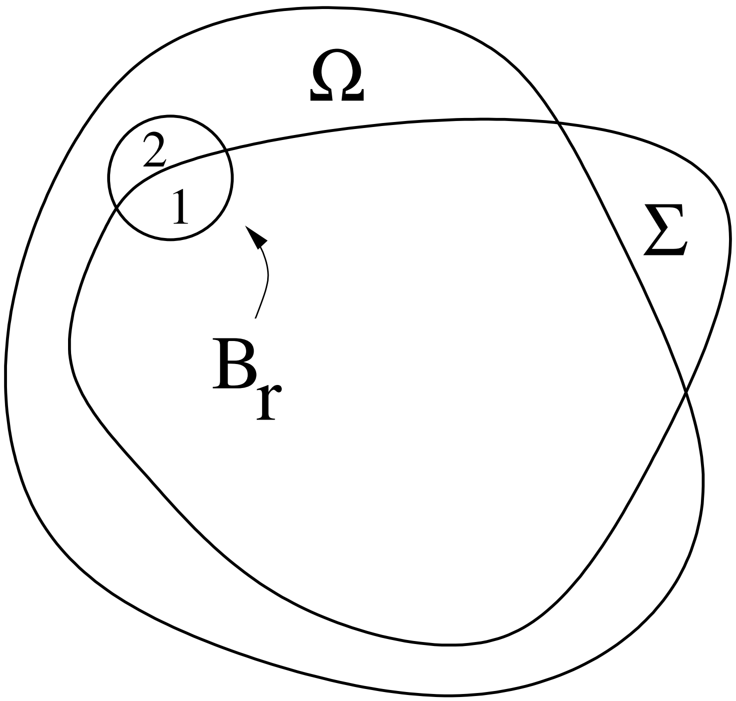


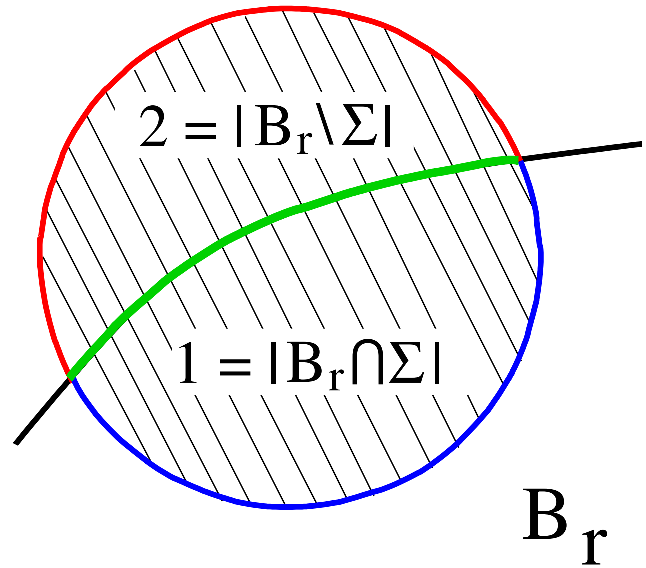


This (plus similar results) permits us to construct nontrivial exact solutions as well as understand non-uniqueness and the precise nature of minimizers.

Geometric Analysis: L^1TV, route to proof for Theorem 1

Comparisons: idea in a picture, using $F(u) = \mathrm{Per}(\Sigma) + \lambda \mathrm{Area}(\Sigma \triangle \Omega)$





A bit more about the team

2001 Started with a 9/11 Homeland Defense LDRD grant,

2002 Organized the 2002 LANL Image Analysis Workshop,

2002-2003: The ECA DR Proposals We eventually get 1 ER and pieces of 2 DR's. Part of the original DR ECA team also gets a spinoff ER on plume detection (Theiler, PI),

2003-present Now rapidly spinning up a virtual team with very strong (in fact integral) external ties.

Our team, currently:

Core 7 LANL, 7 Academic Members

Associates 10-20 internal and external members, depending on how you count.

A Sample of Our Current Projects:

PRAD reconstruction of proton radiographic data

Muon Use of background radiation to find hidden nuclear material

X-ray development of a TVAbel capability for X-ray

Invariant Recognition Ignoring differences that don't matter

Warp Metrics ... and more generally, nonlinear splitting.

MRGA Data consulting and Geometric Analysis.

Mathematical Advances ... in all these efforts.

IPAM Organizing the 2005 IPAM graduate summer school on "Intelligent Extraction of Information from Graphs and High Dimensional Data" : `http://www.ipam.ucla.edu/programs/gss2005/`

IPAM Sponsoring RIPS: `http://www.ipam.ucla.edu/programs/rips2005/`

Final Example: Esedoglu's Bar codes via TV deconvolution